

## ALGEBRAIC VERSUS TOPOLOGICAL TRIANGULATED CATEGORIES

STEFAN SCHWEDE

These are extended and updated notes of a talk, the first version of which I gave at the *Workshop on Triangulated Categories* at the University of Leeds, August 13-19, 2006. These notes are mostly expository and do not contain all proofs; I intend to publish the remaining details elsewhere.

The most commonly known triangulated categories arise from chain complexes in an abelian category by passing to chain homotopy classes or inverting quasi-isomorphisms. Such examples are called ‘algebraic’ because they originate from abelian (or at least additive) categories. Stable homotopy theory produces examples of triangulated categories by quite different means, and in this context the source categories are usually very ‘non-additive’ before passing to homotopy classes of morphisms. Because of their origin I refer to these examples as ‘topological triangulated categories’.

In this note I want to explain some systematic differences between these two kinds of triangulated categories. There are certain properties – defined entirely in terms of the triangulated structure – which hold in all algebraic examples, but which fail in some topological ones. These differences are all torsion phenomena, and rationally there is no difference between algebraic and topological triangulated categories.

A triangulated category is *algebraic* in the sense of Keller [Ke, 3.6] if it is triangle equivalent to the stable category of a Frobenius category, i.e., an exact category with enough injectives and enough projectives in which injectives and projectives coincide. Examples include all triangulated categories which one should reasonably think of as ‘algebraic’: various homotopy categories and derived categories of rings, schemes and abelian categories; stable module categories of Frobenius rings; derived categories of modules over differential graded algebras and differential graded categories. By a theorem of Porta [Po, Thm. 1.2], every algebraic triangulated category which is *well generated* (a mild restriction on its ‘size’, see [Ne, Def. 8.1.6 and 8.1.7]) is equivalent to a localization of the derived category  $\mathcal{D}(\mathcal{A})$  of a small differential graded category  $\mathcal{A}$ .

For an object  $X$  of a triangulated category  $\mathcal{T}$  and a natural number  $n$  we write  $n \cdot \text{Id}_X$  or simply  $n \cdot X$  for the  $n$ -fold multiple of the identity morphism in the group  $[X, X]$  of endomorphisms in  $\mathcal{T}$ . We let  $X/n$  denote any cone of  $n \cdot \text{Id}_X$ , i.e., any object which is part of a distinguished triangle

$$X \xrightarrow{n} X \longrightarrow X/n \longrightarrow X[1] .$$

A short diagram chase shows that the group  $[X/n, X/n]$  is always annihilated by  $n^2$ ; in algebraic triangulated categories, more is true:

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**Proposition 1.** *If  $\mathcal{T}$  is algebraic, then  $n \cdot X/n = 0$ .*

*Proof.* We exploit that algebraic triangulated categories are tensored over  $\mathcal{D}^b(\mathbb{Z})$ , the bounded derived category of finitely generated abelian groups. This means that there is a biexact functor

$$\otimes^L : \mathcal{D}^b(\mathbb{Z}) \times \mathcal{T} \longrightarrow \mathcal{T}$$

which is associative and unital up to coherent isomorphism with respect to the derived tensor product in  $\mathcal{D}^b(\mathbb{Z})$ . In  $\mathcal{D}^b(\mathbb{Z})$  we have a distinguished triangle

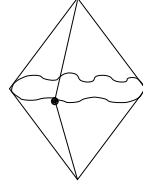
$$\mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow \mathbb{Z}/n \longrightarrow \mathbb{Z}[1]$$

which becomes a distinguished triangle in  $\mathcal{T}$  after tensoring with  $X$ . So  $\mathbb{Z}/n \otimes^L X$  is isomorphic to  $X/n$ . Since the tensor product is additive in each variable and since  $n \cdot \mathbb{Z}/n = 0$  in  $\mathcal{D}^b(\mathbb{Z})$ , we conclude that  $n \cdot (\mathbb{Z}/n \otimes^L X) = 0$ .  $\square$

In contrast to Proposition 1, in a general triangulated category we can have  $n \cdot X/n \neq 0$  for suitable choices of  $X$  and  $n$ . An example arises in the *Spanier-Whitehead category* which we now review. The Spanier-Whitehead category is made from homotopy classes of continuous maps between certain kinds of topological spaces, and it is the prime example of a *topological* triangulated category (to be defined below) which is not algebraic. The Spanier-Whitehead category was originally introduced (without the formal desuspensions) in [SW].

We recall that the *reduced suspension*  $\Sigma X$  of a space  $X$  with basepoint  $x_0$  is given by

$$\Sigma X = \frac{X \times [0, 1]}{X \times \{0, 1\} \cup \{x_0\} \times [0, 1]} .$$



For example we have  $\Sigma S^{n-1} \cong S^n$ , i.e., the suspension of a sphere is homeomorphic to a sphere of the next dimension.

**Definition 2.** The *Spanier-Whitehead category*, denoted  $\mathcal{SW}$ , has as objects the pairs  $(X, n)$  where  $X$  is a finite CW-complex equipped with a distinguished basepoint and  $n \in \mathbb{Z}$ . Morphisms are given by

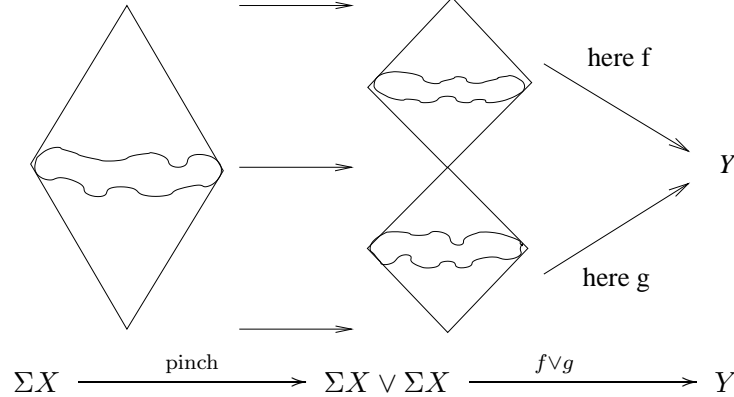
$$\mathcal{SW}((X, n), (Y, m)) = \operatorname{colim}_{k \rightarrow \infty} [\Sigma^{k+n} X, \Sigma^{k+m} Y]$$

where square brackets  $[-, -]$  denote pointed homotopy classes of continuous, basepoint preserving maps. The colimit is formed by iterated suspensions; by Freudenthal's suspension theorem, it is actually attained at a finite stage. Composition in  $\mathcal{SW}$  is defined by composition of representatives, suitably suspended so that composition is possible.

It is convenient to identify a finite pointed CW-complex  $X$  with the object  $(X, 0)$  of the Spanier-Whitehead category. Then two CW-complexes become isomorphic in  $\mathcal{SW}$  if and only if they become homotopy equivalent after a finite number of suspensions.

The Spanier-Whitehead category is triangulated in the following way:

- The shift functor is given by  $(X, n)[1] = (X, n+1)$ . Tautologically, the identity map of  $\Sigma^{n+1}X$  is an isomorphism between  $(\Sigma X, n)$  and  $(X, n+1)$  in  $\mathcal{SW}$ , so suspension is in fact isomorphic to the shift and is invertible in  $\mathcal{SW}$ .
- The Spanier-Whitehead category is additive: for pointed spaces  $X$  and  $Y$ , the set  $[\Sigma X, Y]$  of pointed homotopy classes from a suspension has a natural group structure as follows. The product of  $f, g : \Sigma X \longrightarrow Y$  is represented by the composite



For  $X = S^0$  we have  $\Sigma X \cong S^1$  and this reduces to the group structure on the fundamental group  $[S^1, Y] = \pi_1(Y, y_0)$ . On a double suspension as source object, this group structure is abelian; an example of this is that for  $n \geq 2$  the higher homotopy groups  $\pi_n(Y, y_0) = [S^n, Y]$  are abelian. In the Spanier-Whitehead category, every object is a double suspension, so the homomorphism sets in  $\mathcal{SW}$  are naturally abelian groups.

- Mapping cone sequences give distinguished triangles: the *mapping cone* of a pointed map  $f : X \longrightarrow Y$  is the space

$$C(f) = \frac{X \times [0, 1] \cup_{X \times \{1\}} Y}{X \times \{0\} \cup \{x_0\} \times [0, 1]}$$

There is an inclusion  $i : Y \longrightarrow C(f)$  and a projection  $p : C(f) \longrightarrow \Sigma X$  (which collapses  $Y$  to a point). In the sequence of pointed maps

$$X \xrightarrow{f} Y \xrightarrow{i} C(f) \xrightarrow{p} \Sigma X$$

the composite of any two is null-homotopic, and these sequences and their (de-)suspensions give the distinguished triangles in  $\mathcal{SW}$ .

A verification of the axioms of a triangulated category (except for the octahedral axiom), and more details on the Spanier-Whitehead category can be found in Ch. 1, §2 of Margolis book [Mar]. (Margolis does not impose any finiteness restriction on the objects of the Spanier-Whitehead category; the category  $\mathcal{SW}$  is denoted  $\mathbf{SW}_f$  in [Mar].)

The group

$$\mathcal{SW}(S^n, X) = \operatorname{colim}_{k \rightarrow \infty} [S^{k+n}, \Sigma^k X] = \operatorname{colim}_k \pi_{k+n}(\Sigma^k X, *)$$

is denoted by  $\pi_n^s X$  and called the  $n$ -th *stable homotopy group* of  $X$ . For  $X = S^0$ , we abbreviate  $\pi_n^s S^0$  to  $\pi_n^s$  and speak about the  $n$ -th *stable homotopy group of spheres*, also called the  $n$ -th *stable stem*. For example,  $\pi_0^s = \mathbb{Z}$ , generated by the identity map, and  $\pi_1^s = \mathbb{Z}/2$  generated by the class of the Hopf map  $\eta : S^3 \rightarrow S^2$ . The stable stems are easy to define, but notoriously hard to compute; for example, there is no finite CW-complex  $X$  which is non-trivial in  $\mathcal{SW}$  for which all stable homotopy groups are known! Much machinery of algebraic topology has been developed to calculate such groups and understand their structure, but no one expects to ever get explicit formulae for all stable homotopy groups of spheres.

We also need the symmetric monoidal *smash product* on the Spanier-Whitehead category. This arises from the geometric smash product  $X \wedge Y$  of two pointed spaces  $X$  and  $Y$  defined as  $X \wedge Y = (X \times Y) / (X \times \{y_0\} \cup \{x_0\} \times Y)$ . Suspension is an example of the smash product, i.e.,  $\Sigma X$  is naturally homeomorphic to  $X \wedge S^1$ . The smash product can be extended to the Spanier-Whitehead category by  $(X, n) \wedge (Y, m) = (X \wedge Y, n + m)$ , and then it becomes biexact, i.e., an exact functor of triangulated categories in each variable.

We denote by  $S = (S^0, 0)$  the unit object of the smash product in  $\mathcal{SW}$  and refer to it as the *sphere spectrum*. We use this terminology because the Spanier-Whitehead category can be identified with the compact objects in a larger triangulated category, the *stable homotopy category*, whose objects are called spectra. However, the differences between algebraic and topological triangulated categories already show up in the Spanier-Whitehead category, which is easier to define than the full stable homotopy category.

For  $n \geq 2$ , the *mod- $n$  Moore spectrum* is defined as a cone of multiplication by  $n$  on the sphere spectrum, i.e., it is part of a distinguished triangle

$$(3) \quad S \xrightarrow{n} S \rightarrow S/n \rightarrow S[1].$$

More concretely we can define  $S/n = (S^1 \cup_n D^2, -1)$ , the formal desuspension of a two-dimensional mod- $n$  Moore space, i.e., the space obtained from the circle  $S^1$  by attaching a 2-disc along the degree  $n$  map  $S^1 \rightarrow S^1, z \mapsto z^n$ . For example,  $S/2 = (\mathbb{R}P^2, -1)$ , the formal desuspension of 2-dimensional real projective space. If  $n$  and  $m$  are coprime, then  $S/(nm)$  is isomorphic in the Spanier-Whitehead category to the smash product  $S/n \wedge S/m$ . So we often concentrate on Moore spectra for primes or prime powers, since the other Moore spectra are smash products of these.

Proposition 1 and the following proposition show that the Spanier-Whitehead category is not algebraic.

**Proposition 4.** *The morphism  $2 \cdot S/2$  is nonzero in  $\mathcal{SW}$ .*

The standard tool for proving Proposition 4 and its generalizations below are cohomology operations, which we quickly review. The *mod- $p$  cohomology* of an object  $(X, n)$  in  $\mathcal{SW}$  defined by

$$H^k((X, n), \mathbb{F}_p) = \tilde{H}^{k-n}(X, \mathbb{F}_p)$$

where the right hand side is reduced singular cohomology with coefficients in the field  $\mathbb{F}_p$ . These cohomology groups have a natural action (essentially by definition) by the mod- $p$  *Steenrod algebra*, i.e., the algebra of stable, natural, graded mod- $p$  cohomology operations. The Steenrod algebra is generated for  $p = 2$  by operations  $Sq^i$  of degree  $i$  for  $i \geq 1$  and for odd  $p$  by the Bockstein operation  $\beta$  of degree 1 and operations  $P^i$  of degree  $i(2p - 2)$

for  $i \geq 1$ . For  $p = 2$ , the operation  $Sq^1$  equals the Bockstein operation. These operations satisfy the *Adem relations*, which we do not reproduce here.

The mod- $n$  Moore spectrum is characterized up to isomorphism in the Spanier-Whitehead category by the property that its integral spectrum homology is concentrated in dimension zero where it is isomorphic to  $\mathbb{Z}/n$ . For a prime  $p$  the mod- $p$  cohomology of  $S/p$  is one-dimensional in dimensions 0 and 1, and trivial otherwise, and the Bockstein operation is non-trivial from dimension 0 to dimension 1.

*Proof of Proposition 4.* This proposition is a classical fact, which should probably be credited to Steenrod since the standard proof uses mod-2 Steenrod operations. We argue by contradiction and suppose that  $2 \cdot S/2 = 0$ . If we smash the defining triangle (3) with another copy of the mod-2 Moore spectrum and use that  $S$  is the unit of the smash product, we obtain a distinguished triangle

$$S/2 \xrightarrow{2 \cdot} S/2 \longrightarrow S/2 \wedge S/2 \longrightarrow S/2[1] .$$

Under our assumption the first map is trivial so the smash product  $S/2 \wedge S/2$  splits in  $SW$  as a sum of  $S/2$  and  $S/2[1]$ . Thus as a module over the Steenrod algebra, the mod-2 cohomology of the smash product  $S/2 \wedge S/2$  decomposes into a sum of two non-trivial summands.

On the other hand, there is a Künneth isomorphism for the mod-2 cohomology of a smash product

$$H^n(X \wedge Y, \mathbb{F}_2) \cong \bigoplus_{p+q=n} H^p(X, \mathbb{F}_2) \otimes H^q(Y, \mathbb{F}_2) .$$

This is an isomorphism as modules over the Steenrod algebra with action on the right hand side given by the *Cartan formula*

$$Sq^m(x \otimes y) = \sum_{i=0}^m Sq^i x \otimes Sq^{m-i} y .$$

In the mod-2 cohomology of the mod-2 Moore spectrum the Bockstein operation  $Sq^1 : H^0(S/2, \mathbb{F}_2) \longrightarrow H^1(S/2, \mathbb{F}_2)$  is non-zero. The Cartan formula shows that the operation  $Sq^2$  is then non-trivial on the tensor product of two copies of the generator of  $H^0(S/2, \mathbb{F}_2)$ . Thus the mod-2 cohomology of  $S/2 \wedge S/2$  is a 4-dimensional, indecomposable module over the Steenrod algebra. We have reached a contradiction, which means that we must have  $2 \cdot S/2 \neq 0$  in  $SW$ .  $\square$

In topological triangulated categories, the phenomenon that we can have  $n \cdot X/n \neq 0$  is entirely 2-local. To explain this, I have to be more precise about what I mean by a topological triangulated category. A *model category* (in the sense of Quillen [Q]) is an axiomatic framework for homotopy theoretic constructions. Among other things, a model category structure allows one to define mapping cones and suspensions and talk about homotopies between morphisms. A model category is called *stable* if it has a zero object and the suspension functor is a self-equivalence of its homotopy category. The homotopy category of a stable model category is naturally a triangulated category (cf. [Ho1, 7.1.6]); the proof is essentially the same as for the Spanier-Whitehead category. By definition the suspension functor is a self-equivalence, and it defines the shift functor. Since every object is a two-fold suspension, hence an abelian co-group object, the homotopy category of a

stable model category is additive. The distinguished triangles are defined by mapping cone sequences.

For us a *topological triangulated category* is any triangulated category which is equivalent to a full triangulated subcategory of the homotopy category of a stable model category. An important example which was already mentioned above is the *stable homotopy category* of algebraic topology which was first introduced by Boardman (unpublished; accounts of Boardman's construction appear in [Vo] and [Ad, Part III]). There is an abundance of models for the stable homotopy category, see for example [BF, EKMM, HSS, MMSS, Ly]. The Spanier-Whitehead category  $\mathcal{SW}$  is equivalent to the full subcategory of compact objects in the stable homotopy category, so it is a topological triangulated category in our sense. Further examples of topological triangulated categories are 'derived' (i.e., homotopy) categories of structured ring spectra, equivariant and motivic stable homotopy categories, sheaves of spectra on a Grothendieck site or (Bousfield-) localizations of all these, see [SS, Sec. 2.3] for more details. The theorem of Porta mentioned above has an analogue in this context: Heider essentially shows in [He, Thm. 4.7] that every topological triangulated category which is well generated is equivalent to a localization of the homotopy category  $\mathrm{Ho}(\mathcal{R}\text{-mod})$  of a small spectral category  $\mathcal{R}$ .

Algebraic triangulated categories are typically also topological (the converse is not generally true, and that is the point of these notes). For algebraic triangulated categories which are derived categories of abelian categories, this follows whenever there is a model structure on the category of chain complexes with quasi-isomorphisms as weak equivalences (see for example [Ho2] or Section 2.4 of [SS] for more details and references). Similarly, for modules over a Frobenius ring, there is a stable model structure with stable equivalences as the weak equivalences, see [Ho1, Thm. 2.2.12]. More generally, any well-generated algebraic triangulated category is equivalent to a localization of the derived category  $\mathcal{D}(\mathcal{A})$  of small differential graded category  $\mathcal{A}$  [Po, Thm. 1.2]. The localization can be realized as a Bousfield localization of the ordinary (i.e., 'projective') model structure on modules over  $\mathcal{A}$ ; hence  $\mathcal{D}(\mathcal{A})$  and its localizations are topological.

Examples of triangulated categories which are neither algebraic nor topological were recently constructed by Muro, Strickland and the author [MSS]. The simplest one is the category  $\mathcal{F}(\mathbb{Z}/4)$  of finitely generated free modules over the ring  $\mathbb{Z}/4$ . The category  $\mathcal{F}(\mathbb{Z}/4)$  has a unique triangulation with the identity shift functor and such that the triangle

$$\mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4$$

is exact. For the proof of this and an argument why  $\mathcal{F}(\mathbb{Z}/4)$  is not topological I refer to [MSS]. At present, I do not know any 'exotic' (i.e., non-topological and non-algebraic) triangulated category in which 2 is invertible.

Unlike algebraic triangulated categories, we can not usually expect that a topological triangulated category can be tensored over  $\mathcal{D}^b(\mathbb{Z})$ , the bounded derived category of finitely generated abelian groups. The appropriate replacement for  $\mathcal{D}^b(\mathbb{Z})$  is the Spanier-Whitehead category: for every topological triangulated category  $\mathcal{T}$ , there is a biexact pairing

$$\wedge : \mathcal{SW} \times \mathcal{T} \longrightarrow \mathcal{T}$$

which is associative and unital up to coherent natural isomorphism with respect to the smash product in the Spanier-Whitehead category. The shift functor in  $\mathcal{T}$  is isomorphic to smashing with the circle  $S^1$ , view as an object of  $\mathcal{SW}$ .

We use the above pairing as a black box, but I'll briefly indicate how it can be constructed. One way is to start from the action of the homotopy category of pointed simplicial sets on the homotopy category of a pointed model category  $\mathcal{C}$ ,

$$\wedge : \mathrm{HoSSet}_* \times \mathrm{Ho} \mathcal{C} \longrightarrow \mathrm{Ho} \mathcal{C} ,$$

which uses a technique called ‘framings’, see [Ho1, Thm. 5.7.3]. The geometric realization of a finite simplicial set is a finite CW-complex, and up to homotopy equivalence, every finite CW-complex arises in this way. Since moreover suspension is invertible in  $\mathrm{Ho} \mathcal{C}$ , this extends to a well-defined pairing on the Spanier-Whitehead category.

There is an alternative construction for *spectral model categories*, i.e., model categories  $\mathcal{C}$  which are enriched over the category  $Sp^\Sigma$  of symmetric spectra of [HSS], compatibly with the smash product and the stable model structure (see [SS, Def. 3.5.1] for details). The derived smash product between symmetric spectra and  $\mathcal{C}$  descends to an associative and unital pairing

$$\wedge : \mathrm{Ho}(Sp^\Sigma) \times \mathrm{Ho} \mathcal{C} \longrightarrow \mathrm{Ho} \mathcal{C} ,$$

which can be restricted to the full subcategory  $\mathcal{SW}$  of compact objects in  $\mathrm{Ho}(Sp^\Sigma)$ . A general stable model category is Quillen-equivalent to a spectral model category, under mild technical hypothesis (see [SS, Thm. 3.8.2], [Du, Prop. 5.5 (a) and 5.6 (a)] and [Ho3, Thm. 9.1 and 8.11] for different sets of sufficient conditions).

In algebraic examples the action of the Spanier-Whitehead category  $\mathcal{SW}$  and the bounded derived category  $\mathcal{D}^b(\mathbb{Z})$  (which we used in the proof of Proposition 1) are related as follows. The chain functor  $C_* : \mathcal{SW} \longrightarrow \mathcal{D}^b(\mathbb{Z})$  associates to an object  $(X, n)$  of the Spanier-Whitehead category the reduced singular chain complex of the CW-complex  $X$ , shifted up  $n$  dimensions. This chain functor is strong symmetric monoidal, i.e., there are associative, unital and commutative isomorphisms  $C_*(X \wedge Y) \cong C_*(X) \otimes^L C_*(Y)$  in  $\mathcal{D}^b(\mathbb{Z})$ . If  $\mathcal{T}$  is an algebraic triangulated category which is also topological, then the composite

$$\mathcal{SW} \times \mathcal{T} \xrightarrow{C_* \times \mathrm{Id}} \mathcal{D}^b(\mathbb{Z}) \times \mathcal{T} \xrightarrow{\otimes^L} \mathcal{T}$$

is naturally isomorphic to the smash product pairing.

Now we can make precise in which sense the possibility of having  $n \cdot X/n \neq 0$  in topological triangulated categories is a 2-local phenomenon.

**Proposition 5.** *If  $\mathcal{T}$  is a topological triangulated category and  $p$  an odd prime, then  $p \cdot X/p = 0$  for every object  $X$  of  $\mathcal{T}$ .*

*Proof.* The key point is that if  $n \not\equiv 2 \pmod{4}$ , then the mod- $n$  Moore spectrum  $S/n$  is annihilated by  $n$  (which is not the case for  $n = 2$ ). We recall the easy proof for odd  $n$ , which includes all odd primes. If we take homotopy groups of the defining triangle (3) (i.e., apply  $[S, -]$ , where  $[-, -]$  denotes morphisms in  $\mathcal{SW}$ ) we obtain an exact sequence

$$\pi_1^s \xrightarrow{n} \pi_1^s \longrightarrow \pi_1(S/n) \longrightarrow \pi_0^s \xrightarrow{n} \pi_0^s \longrightarrow \pi_0(S/n) \longrightarrow 0 ,$$

using that the stable homotopy groups of spheres vanish in negative dimensions. Since  $\pi_0^s \cong \mathbb{Z}$  and  $\pi_1^s \cong \mathbb{Z}/2$  we deduce that  $\pi_0(S/n)$  is cyclic of order  $n$  and  $\pi_1^s(S/n) = 0$  for

odd  $n$ . Now we apply  $[-, S/n]$  to the triangle (3) and we obtain a short exact sequence

$$0 \longrightarrow [S[1], S/n] \otimes \mathbb{Z}/n \longrightarrow [S/n, S/n] \longrightarrow {}_n[S, S/n] \longrightarrow 0$$

where  ${}_nA = \{a \in A \mid na = 0\}$  denotes the group of  $n$ -torsion points in an abelian group  $A$ . By the above calculations, the group  $[S[1], S/n] \otimes \mathbb{Z}/n$  vanishes and so the group  $[S/n, S/n]$  is cyclic of order  $n$  for  $n$  odd, hence  $n \cdot S/n = 0$ . (For  $n = 2$  we have  $\pi_1(S/2) \cong \mathbb{Z}/2$  and the analogous short exact sequence does not split by Proposition 4. So  $[S/2, S/2] \cong \mathbb{Z}/4$ .)

The rest of the argument is then the same as in Proposition 1. For an odd prime  $p$  we can smash the distinguished triangle

$$S \xrightarrow{p} S \longrightarrow S/p \longrightarrow S[1]$$

in  $\mathcal{SW}$  with the object  $X$  and obtain a distinguished triangle in  $\mathcal{T}$  which shows that  $S/p \wedge X$  is isomorphic to  $X/p$ . Since  $p \cdot S/p = 0$  in  $\mathcal{SW}$  and  $\wedge$  is biadditive, we conclude that  $p \cdot X/p = p \cdot (S/p \wedge X) = 0$ .  $\square$

We have seen in Proposition 4 that  $p \cdot X/p$  can be non-zero for  $p = 2$ . On the other hand, all triangulated categories that I know have the property that  $p \cdot X/p = 0$  for odd primes  $p$  and all objects  $X$ . This leaves us with the

**Open problem 6.** Let  $p$  be an odd prime. Find a triangulated category  $\mathcal{T}$  and an object  $X$  of  $\mathcal{T}$  such that  $p \cdot X/p \neq 0$ , or prove that in every triangulated category  $\mathcal{T}$  we always have  $p \cdot X/p = 0$ .

From what we have discussed so far, it is still conceivable that every topological triangulated category in which 2 is invertible is algebraic. In particular one can wonder whether the Spanier-Whitehead category is algebraic after localization at an odd prime. We now describe a property of triangulated categories which distinguishes topological from algebraic examples away from the prime 2.

As before we denote by  $K/n$  any cone of  $n \cdot \text{Id}_K$ , which comes as part of a distinguished triangle

$$K \xrightarrow{n} K \xrightarrow{\pi} K/n \longrightarrow K[1].$$

An *extension* of a morphism  $f : K \longrightarrow Y$  is then a morphism  $\bar{f} : K/n \longrightarrow Y$  satisfying  $\bar{f}\pi = f$ . Such an extension exists if and only if  $n \cdot f = 0$ , and then the extension will usually not be unique.

**Proposition 7.** *Let  $\mathcal{T}$  be an algebraic triangulated category,  $X$  an object of  $\mathcal{T}$  and  $n \geq 2$ . Then every morphism  $f : K \longrightarrow X/n$  has an extension  $\bar{f} : K/n \longrightarrow X/n$  such that some (hence any) mapping cone of  $\bar{f}$  is annihilated by  $n$ .*

*Proof.* As in Proposition 1, a choice of model for  $\mathcal{T}$  as the stable category of a Frobenius category gives a biexact, associative and unital pairing  $\otimes^L : \mathcal{D}^b(\mathbb{Z}) \times \mathcal{T} \longrightarrow \mathcal{T}$  and we can take  $X/n = \mathbb{Z}/n \otimes^L X$  and  $K/n = \mathbb{Z}/n \otimes^L K$ . We define the extension  $\bar{f}$  as the composite

$$\mathbb{Z}/n \otimes^L K \xrightarrow{\mathbb{Z}/n \otimes f} \mathbb{Z}/n \otimes^L \mathbb{Z}/n \otimes^L X \xrightarrow{\mu \otimes X} \mathbb{Z}/n \otimes^L X$$

where  $\mu : \mathbb{Z}/n \otimes^L \mathbb{Z}/n \longrightarrow \mathbb{Z}/n$  is the multiplication map which makes  $\mathbb{Z}/n$  into a ring. We choose a distinguished triangle

$$(8) \quad \mathbb{Z}/n \otimes^L K \xrightarrow{\bar{f}} \mathbb{Z}/n \otimes^L X \xrightarrow{\varphi} C(\bar{f}) \xrightarrow{\delta} \mathbb{Z}/n \otimes^L K[1]$$



and show that the mapping cone  $C(\bar{f})$  of  $\bar{f}$  is annihilated by  $n$ .

We consider the diagram

$$\begin{array}{ccccccc}
\mathbb{Z}/n \otimes^L \mathbb{Z}/n \otimes^L K & \xrightarrow{\mathbb{Z}/n \otimes \bar{f}} & \mathbb{Z}/n \otimes^L \mathbb{Z}/n \otimes^L X & \xrightarrow{\mathbb{Z}/n \otimes \varphi} & \mathbb{Z}/n \otimes^L C(\bar{f}) & \xrightarrow{\mathbb{Z}/n \otimes \delta} & \mathbb{Z}/n \otimes^L \mathbb{Z}/n \otimes^L K[1] \\
\mu \otimes K \downarrow & & \downarrow \mu \otimes X & & \downarrow \sigma & & \downarrow \mu \otimes K[1] \\
\mathbb{Z}/n \otimes^L K & \xrightarrow{\bar{f}} & \mathbb{Z}/n \otimes^L X & \xrightarrow{\varphi} & C(\bar{f}) & \xrightarrow{\delta} & \mathbb{Z}/n \otimes^L K[1]
\end{array}$$

whose lower row is the distinguished triangle (8) and whose upper row is (8) tensored from the left with  $\mathbb{Z}/n$ . The left square commutes since the multiplication morphism  $\mu$  is associative in  $\mathcal{D}^b(\mathbb{Z})$ . Since both rows are distinguished triangles, there exists a morphism  $\sigma : \mathbb{Z}/n \otimes^L C(\bar{f}) \rightarrow C(\bar{f})$  making the middle and right square commute.

We consider the morphism

$$\pi \otimes C(\bar{f}) : C(\bar{f}) \cong \mathbb{Z} \otimes^L C(\bar{f}) \rightarrow \mathbb{Z}/n \otimes^L C(\bar{f})$$

which satisfies the two relations

$$(9) \quad \sigma(\pi \otimes C(\bar{f}))\varphi = \sigma(\mathbb{Z}/n \otimes \varphi)(\pi \otimes \mathbb{Z}/n \otimes X) = \varphi(\mu \otimes X)(\pi \otimes \mathbb{Z}/n \otimes X) = \varphi$$

and

$$\begin{aligned}
(10) \quad \delta\sigma(\pi \otimes C(\bar{f})) &= (\mu \otimes K[1])(\mathbb{Z}/n \otimes \delta)(\pi \otimes C(\bar{f})) \\
&= (\mu \otimes K[1])(\pi \otimes \mathbb{Z}/n \otimes K[1])\delta = \delta.
\end{aligned}$$

We claim that the morphism

$$\sigma' = 2\sigma - \sigma(\pi \otimes C(\bar{f}))\sigma : \mathbb{Z}/n \otimes^L C(\bar{f}) \rightarrow C(\bar{f})$$

is a retraction to  $\pi \otimes C(\bar{f})$ . Indeed, by (9) the composite of  $\varphi : \mathbb{Z}/n \otimes^L X \rightarrow C(\bar{f})$  with the morphism  $\sigma(\pi \otimes C(\bar{f})) - \text{Id} : C(\bar{f}) \rightarrow C(\bar{f})$  becomes trivial, so there exists a morphism  $g : \mathbb{Z}/n \otimes^L K[1] \rightarrow C(\bar{f})$  such that  $g\delta = \sigma(\pi \otimes C(\bar{f})) - \text{Id}$ . But then we have

$$\begin{aligned}
\sigma'(\pi \otimes C(\bar{f})) &= 2\sigma(\pi \otimes C(\bar{f})) - (\text{Id} + g\delta)\sigma(\pi \otimes C(\bar{f})) \\
&= \sigma(\pi \otimes C(\bar{f})) - g\delta\sigma(\pi \otimes C(\bar{f})) = \sigma(\pi \otimes C(\bar{f})) - g\delta = \text{Id},
\end{aligned}$$

as claimed, where the third equality uses (10).

Now we have shown that  $C(\bar{f})$  is a direct summand of  $\mathbb{Z}/n \otimes^L C(\bar{f})$ , which is annihilated by  $n$ , and so we have  $n \cdot C(\bar{f}) = 0$ .  $\square$

In Theorem 16 below we prove a generalization of Proposition 7, by a different method.

Here is an example showing that in the situation of Proposition 7 in general there may not exist any extension  $\bar{f} : K/n \rightarrow X/n$  of  $f$  whose mapping cone is annihilated by  $n$ . Proposition 7 and the following proposition show that the Spanier-Whitehead category localized at 3 is not algebraic.

We let  $\beta_1 \in \pi_{10}^s \cong \mathbb{Z}/6$  be an element of order 3 (so  $\beta_1$  generates the 3-primary component of the 10-dimensional stable stem). One way to define  $\beta_1$  is as the unique element of the Toda bracket  $\langle \alpha_1, \alpha_1, \alpha_1 \rangle$  where  $\alpha_1 \in \pi_3^s \cong \mathbb{Z}/24$  is the 3-primary part of the element represented by the second Hopf map  $\nu : S^7 \rightarrow S^4$ . We let  $\tilde{\beta}_1 : S[11] \rightarrow S/3$  be any lift

in the Spanier-Whitehead category  $\mathcal{SW}$  of  $\beta_1$  to the mod-3 Moore spectrum  $S/3$ . So  $\tilde{\beta}_1$  is a morphism whose composite with the connecting map  $S/3 \rightarrow S[1]$  equals the shift of  $\beta_1$ . We will need below that any such lift  $\tilde{\beta}_1$  is *detected by*  $P^3$  in the sense that in any mapping cone of  $\tilde{\beta}_1$  the Steenrod operation  $P^3$  is a non-trivial isomorphism from mod-3 cohomology in dimension 0 to dimension 12 (see page 60 of [To, §5]).

**Proposition 11.** *There is no extension of  $\tilde{\beta}_1$  to a morphism  $\bar{\beta} : S/3[11] \rightarrow S/3$  whose mapping cone is annihilated by 3.*

*Proof.* We argue by contradiction and suppose that there exists an extension

$$\bar{\beta} : S/3[11] \rightarrow S/3$$

of  $\tilde{\beta}_1$  and a distinguished triangle

$$S/3[11] \xrightarrow{\bar{\beta}} S/3 \rightarrow C(\bar{\beta}) \xrightarrow{\delta} S/3[12]$$

with  $3 \cdot C(\bar{\beta}) = 0$ .

Since the stable stems in dimension 21 and 22 consist only of torsion prime to 3 we have  $\pi_{22}(S/3) = [S[22], S/3] = 0$ . So there is a morphism  $a : S[23] \rightarrow C(\bar{\beta})$  lifting  $\tilde{\beta}_1$  in the sense that  $\delta a = \tilde{\beta}_1[12]$ . Since we assumed  $3 \cdot C(\bar{\beta}) = 0$ , the morphism  $a$  can be extended to a morphism  $\bar{a} : S/3[23] \rightarrow C(\bar{\beta})$ . We let  $C(\bar{a})$  be a mapping cone of  $\bar{a}$  arising as part of a distinguished triangle

$$S/3[23] \xrightarrow{\bar{a}} C(\bar{\beta}) \rightarrow C(\bar{a}) \xrightarrow{\delta} S/3[24].$$

Since the stable stems in dimension 21, 22, 33 and 34 consist only of torsion prime to 3 we have  $\pi_{34}C(\bar{\beta}) = [S[34], C(\bar{\beta})] = 0$  and so there is a morphism  $b : S[35] \rightarrow C(\bar{a})$  lifting  $\tilde{\beta}_1$  in the sense that  $\delta b = \tilde{\beta}_1[24]$ .

Now we bring cohomology operations into the game to reach a contradiction. The Moore spectrum  $S/3$  has its mod-3 cohomology concentrated in dimensions 0 and 1, where it is 1-dimensional. The mapping cone of  $b$  is built by distinguished triangles from three shifted copies of  $S/3$  and one shifted copy of  $S$ , so its mod-3 cohomology is concentrated in dimensions 0, 1, 12, 13, 24, 25 and 36, where it is 1-dimensional.

The morphism  $\tilde{\beta}_1$  is detected by the Steenrod operation  $P^3$ , i.e., in the mod-3 cohomology of the mapping cone of  $\tilde{\beta}_1$  the operation  $P^3$  is non-trivial from dimension 0 to dimension 12. The stable ‘cells’ (i.e., shifted copies of  $S$ ) of the mapping cone of  $b$  in dimension 12, 24 and 36, are attached to the two cells directly below by  $\tilde{\beta}_1$ , so in the cohomology of the mapping cone of  $b$ , the 3-fold iterate of  $P^3$  is non-trivial from dimension 0 to 36. By the Adem relations we have  $(P^3)^3 = (P^7P^1 - P^8)P^1$ . Since the cohomology is trivial in dimension 4 the operation  $P^1$  acts trivially, hence so does  $(P^3)^3$ . We have obtained a contradiction, and so no extension of  $\tilde{\beta}_1$  has a mapping cone which is annihilated by 3.  $\square$

In topological triangulated categories, the phenomenon that a morphism  $f : K \rightarrow X/n$  may not have any extension whose cone is annihilated by  $n$  is entirely 2- and 3-local, in the following sense.

**Proposition 12.** *If  $\mathcal{T}$  is a topological triangulated category and  $p$  is a prime bigger than 3, then every morphism  $f : K \rightarrow X/p$  has an extension  $\bar{f} : K/p \rightarrow X/p$  whose mapping cone is annihilated by  $p$ .*

*Proof.* The key point is that if  $n$  is prime to 6, then the mod- $n$  Moore spectrum has an associative multiplication in the Spanier-Whitehead category (the multiplication is also commutative, but that is not relevant for the current proof). Again I cannot refrain from giving the simple proof below. In contrast, the mod-2 Moore spectrum does not have a multiplication; the mod-3 Moore spectrum has a commutative multiplication, but that is not associative in  $\mathcal{SW}$ , see [To].

As in the proof of Proposition 5, the condition that  $n$  is odd guarantees that  $n \cdot S/n = 0$ . So there exists a morphism  $\mu : S/n \wedge S/n \longrightarrow S/n$  which splits the two ‘inclusions’. We show that  $\mu$  is associative. In fact, the *associator*

$$\mu(\mu \wedge \text{Id} - \text{Id} \wedge \mu) : S/n \wedge S/n \wedge S/n \longrightarrow S/n$$

factors as a composite

$$S/n \wedge S/n \wedge S/n \xrightarrow{\delta \wedge \delta \wedge \delta} S[3] \longrightarrow S/n$$

where  $\delta : S/n \longrightarrow S[1]$  is the connecting morphism. We have  $\pi_3^s \cong \mathbb{Z}/24$  and  $\pi_2^s \cong \mathbb{Z}/2$ , so if  $n$  is prime to 6 we have  $[S[3], S/n] = \pi_3(S/n) = 0$ . Thus the associator is trivial or, equivalently,  $\mu$  is associative in  $\mathcal{SW}$ .

The rest of the argument is now essentially the same as in Proposition 7. We can smash the distinguished triangle

$$S \xrightarrow{n} S \longrightarrow S/n \xrightarrow{\delta} S[1]$$

in  $\mathcal{SW}$  with the object  $X$  and obtain a distinguished triangle in  $\mathcal{T}$  which shows that  $S/n \wedge X$  is isomorphic to  $X/n$ . We use the multiplication  $\mu : S/n \wedge S/n \longrightarrow S/n$  to define the extension  $\bar{f}$  as the composite

$$S/n \wedge K \xrightarrow{\text{Id} \wedge f} S/n \wedge S/n \wedge X \xrightarrow{\mu \wedge \text{Id}} S/n \wedge X.$$

Then we use the same reasoning as in the proof of Proposition 7 to obtain the retraction  $\sigma' : S/n \wedge C(\bar{f}) \longrightarrow C(\bar{f})$  which shows that  $n \cdot C(\bar{f}) = 0$ .  $\square$

This raises the following question:

**Open problem 13.** Consider a prime  $p \geq 5$ . Does there exist a triangulated category  $\mathcal{T}$ , an object  $X$  of  $\mathcal{T}$  and a morphism  $f : K \longrightarrow X/p$  which does not admit any extension  $\bar{f}$  to  $K/p$  whose mapping cone is annihilated by  $p$ ?

**Remark 14.** The proof of Proposition 12 shows that the special features of topological over algebraic triangulated categories are closely related to existence and properties of multiplications on mod- $n$  Moore spectra. For primes  $p \geq 5$ , the mod- $p$  Moore spectrum has a multiplication in the Spanier-Whitehead category which is commutative and associative. So on the level of tensor triangulated categories, there does not seem to be any qualitative difference between the Moore spectrum  $S/p$  as an object of the Spanier-Whitehead category  $\mathcal{SW}$  and  $\mathbb{Z}/p$  as an object of  $\mathcal{D}^b(\mathbb{Z})$ , as long as  $p \geq 5$ . However, mod- $n$  Moore spectra are never  $A_\infty$  ring spectra, but rigorously defining what that means and proving it would lead us too far afield. Theorem 17 below explains how the higher order non-associativity eventually manifests itself in the triangulated structure of the Spanier-Whitehead category (i.e., without any reference to the smash product).

From what we have discussed so far, it is still conceivable that for primes  $p \geq 5$  the  $p$ -local Spanier-Whitehead category is algebraic. We will now introduce an invariant which we then use to show that this is not the case for any prime  $p$ .

**Definition 15.** Consider a triangulated category  $\mathcal{T}$  and a natural number  $n \geq 2$ . We define the  $n$ -order for objects  $Y$  of  $\mathcal{T}$  inductively.

- Every object has  $n$ -order greater or equal to 0.
- For  $k \geq 1$ , an object  $Y$  has  $n$ -order greater or equal to  $k$  if and only if for every object  $K$  of  $\mathcal{T}$  and every morphism  $f : K \rightarrow Y$  there exists an extension  $\bar{f} : K/n \rightarrow Y$  such that some (hence any) mapping cone of  $\bar{f}$  has  $n$ -order greater or equal to  $k - 1$ .

We write  $n\text{-ord}(Y)$ , or  $n\text{-ord}^{\mathcal{T}}(Y)$  if we need to specify the ambient triangulated category, for the  $n$ -order of  $Y$ , i.e., the largest  $k$  (possibly infinite) such that  $Y$  has  $n$ -order greater or equal to  $k$ . We define the  $n$ -order of the triangulated category  $\mathcal{T}$  as the  $n$ -order of some (hence any) zero object, and denote it by  $n\text{-ord}(\mathcal{T})$ . We make some observations which are direct consequences of the definitions.

- The  $n$ -order for objects is invariant under isomorphism and shift.
- An object  $Y$  has positive  $n$ -order if and only if every morphism  $f : K \rightarrow Y$  has an extension to  $K/n$ , which is equivalent to  $n \cdot f = 0$ . So  $n\text{-ord}(Y) \geq 1$  is equivalent to the condition  $n \cdot Y = 0$ .
- The  $n$ -order of a triangulated category is one larger than the minimum of the  $n$ -orders of all objects of the form  $K/n$ .
- Let  $\mathcal{S} \subseteq \mathcal{T}$  be a full triangulated subcategory and  $Y$  an object of  $\mathcal{S}$ . Induction on  $k$  shows that if  $n\text{-ord}^{\mathcal{T}}(Y) \geq k$ , then  $n\text{-ord}^{\mathcal{S}}(Y) \geq k$ . Thus we have  $n\text{-ord}^{\mathcal{S}}(Y) \geq n\text{-ord}^{\mathcal{T}}(Y)$ . In the special case of a zero object we get  $n\text{-ord}(\mathcal{S}) \geq n\text{-ord}(\mathcal{T})$ .
- Suppose that  $\mathcal{T}$  is a  $\mathbb{Z}[1/n]$ -linear triangulated category, i.e., multiplication by  $n$  is an isomorphism for every object of  $\mathcal{T}$ . Then  $K/n$  is trivial for every object  $K$  and thus  $\mathcal{T}$  has infinite  $n$ -order. If on the other hand  $Y$  is non-trivial, then  $n\text{-ord}(Y) = 0$ .
- If every object of  $\mathcal{T}$  has positive  $n$ -order, then  $n \cdot Y = 0$  for all objects  $Y$  and so  $\mathcal{T}$  is a  $\mathbb{Z}/n$ -linear triangulated category. Suppose conversely that  $\mathcal{T}$  is a  $\mathbb{Z}/n$ -linear triangulated category. Then induction on  $k$  shows that  $n\text{-ord}(Y) \geq k$  for all objects  $Y$ , and thus every object has infinite  $n$ -order.

The last two items show that the  $n$ -order is useless if  $\mathcal{T}$  is a  $k$ -linear triangulated category for some field  $k$ , since then every  $n \in \mathbb{Z}$  is either zero or a unit in  $k$ .

The results which we have obtained so far can be rephrased using the notion of  $n$ -order: if  $\mathcal{T}$  is algebraic, then Propositions 1 and 7 show that for every object  $X$ , the object  $X/n$  always has  $n$ -order at least 2; we will improve this in Theorem 16 below. Propositions 4 and 11 show that in the Spanier-Whitehead category, the Moore spectrum  $S/2$  has 2-order 0 and  $S/3$  has 3-order 1; we generalize this to mod- $p$  Moore spectra in Theorem 17 below. If  $\mathcal{T}$  is topological then for every object  $X$ , the object  $X/3$  has 3-order at least 1, by Proposition 5. If  $\mathcal{T}$  is topological and  $p$  is a prime  $\geq 5$ , then for every object  $X$ , the object  $X/p$  has  $p$ -order at least 2, by Proposition 12.

The following theorem generalizes Propositions 1 and 7.

**Theorem 16.** *Let  $\mathcal{T}$  be an algebraic triangulated category and  $X$  an object of  $\mathcal{T}$ . Then for any  $n \geq 2$ , the object  $X/n$  has infinite  $n$ -order. In particular, every algebraic triangulated category  $\mathcal{T}$  has infinite  $n$ -order.*

*Proof.* (Sketch) The assumption that  $\mathcal{T}$  is algebraic gives another piece of extra structure: for any integer  $n$  there exists a triangulated category  $\mathcal{T}/n$  and an adjoint pair of exact functors  $\rho_* : \mathcal{T} \rightarrow \mathcal{T}/n$  and  $\rho^* : \mathcal{T}/n \rightarrow \mathcal{T}$  such that for every object  $X$  of  $\mathcal{T}$  there exists a distinguished triangle

$$X \xrightarrow{n} X \xrightarrow{\eta} \rho^*(\rho_* X) \rightarrow X[1]$$

where  $\eta$  is the unit of the adjunction.

We do not construct  $\mathcal{T}/n$  in general here, but content ourselves with an example which gives the main idea. If  $\mathcal{T} = \mathcal{D}(A)$  is the derived category of a differential graded ring  $A$ , then we can take  $\mathcal{T}/n$  as the derived category of the differential graded ring  $A \otimes \overline{\mathbb{Z}/n}$ , where  $\overline{\mathbb{Z}/n}$  is a flat resolution of the ring  $\mathbb{Z}/n$ , for example the exterior algebra over  $\mathbb{Z}$  on a generator  $x$  of dimension 1 with differential  $dx = n$ . The adjoint functor pair  $(\rho_*, \rho^*)$  is derived from restriction and extension of scalars along the morphism  $A \rightarrow A \otimes \overline{\mathbb{Z}/n}$ .

Now we exploit the extra structure to prove the theorem. Since  $X/n$  is isomorphic to  $\rho^*(\rho_* X)$  it is enough to show that for every object  $Z$  of  $\mathcal{T}/n$  and all  $k \geq 0$  the object  $\rho^* Z$  has  $n$ -order greater or equal to  $k$ .

We proceed by induction on  $k$ ; for  $k = 0$  there is nothing to prove. Suppose we have already shown that every  $\rho^* Z$  has  $n$ -order greater or equal to  $k - 1$  for some positive  $k$ . Given a morphism  $f : K \rightarrow \rho^* Z$  in  $\mathcal{T}$  we can consider its adjoint  $\hat{f} : \rho_* K \rightarrow Z$  in  $\mathcal{T}/n$ ; if we apply  $\rho^*$  we obtain an extension  $\rho^*(\hat{f}) : K/n \cong \rho^*(\rho_* K) \rightarrow \rho^* Z$  of  $f$ . We choose a cone of  $\hat{f}$ , i.e., a distinguished triangle

$$\rho_* K \xrightarrow{\hat{f}} Z \rightarrow C(\hat{f}) \rightarrow \rho_* K[1]$$

in  $\mathcal{T}/n$ . Since  $\rho^*$  is exact,  $\rho^* C(\hat{f})$  is a cone of the extension  $\rho^*(\hat{f})$  in  $\mathcal{T}$ . By induction,  $\rho^* C(\hat{f})$  has  $n$ -order greater or equal to  $k - 1$ , which proves that  $\rho^* Z$  has  $n$ -order greater or equal to  $k$ .  $\square$

The next theorem generalizes Propositions 4 and 11 and shows that topological triangulated categories behave quite differently from algebraic ones.

**Theorem 17.** *Let  $p$  be a prime. Then in the Spanier-Whitehead category, the mod- $p$  Moore spectrum  $S/p$  has  $p$ -order  $p - 2$ . Moreover, the Spanier-Whitehead category, has  $p$ -order  $p - 1$ .*

The proof of Theorem 17 has two parts. One ingredient is a general statement about  $p$ -orders in *topological* triangulated categories  $\mathcal{T}$  which generalizes Propositions 5 and 12: for any object  $X$  and prime  $p$ , the object  $X/p$  has  $p$ -order greater or equal to  $p - 2$ . The proof of this result uses the concept and properties of a *coherent action* of a mod- $p$  Moore space on an object of a model category, see Section 2 of [Sch]. It follows that any topological triangulated category (such as the Spanier-Whitehead category) has  $p$ -order at least  $p - 1$ .

The second ingredient of Theorem 17 is the proof that the mod- $p$  Moore spectrum  $S/p$  has  $p$ -order at most  $p - 2$ . This uses mod- $p$  cohomology operations and serious calculational

input; in particular, the proof depends on vanishing results, due to other people, about the  $p$ -primary components of the stable stems in specific dimensions. Proposition 11 gives the flavor of the proof which, however, for general primes  $p$  is more involved. I plan to give a detailed proof elsewhere.

As we just mentioned, every topological triangulated category has  $p$ -order at least  $p - 1$ . This leave us with the following question, which generalizes Problems 6 and 13

**Open problem 18.** Let  $p$  be a an odd prime. Does there exist a triangulated category whose  $p$ -order is strictly less than  $p - 1$ ?

More generally we can ask which values the  $n$ -orders of triangulated categories can take.

Now that we have discussed torsion phenomena which can distinguish algebraic from topological triangulated categories, it is natural to ask whether there are any differences between topological and algebraic triangulated categories if all primes are invertible, i.e., in  $\mathbb{Q}$ -linear triangulated categories. The  $n$ -order is rationally a useless invariant since  $\mathbb{Q}$ -linear triangulated categories have infinite  $n$ -order for all  $n$ . Similarly, the smash product pairing of a topological triangulated category with  $\mathcal{SW}$  gives no extra information for  $\mathbb{Q}$ -linear triangulated categories since the chain functor  $C_* : \mathcal{SW} \rightarrow \mathcal{D}^b(\mathbb{Z})$  becomes an equivalence of categories when rationalized (both sides are in fact rationally equivalent to the category of finite dimensional graded  $\mathbb{Q}$ -vector spaces).

It turns out that rationally the notions of algebraic and topological triangulated categories essentially coincide. Under some mild technical assumptions and cardinality restriction, every  $\mathbb{Q}$ -linear topological triangulated category is algebraic. More precisely, a theorem of Shipley [Sh, Cor. 2.16] says that every  $\mathbb{Q}$ -linear *spectral* model category (a special kind of stable model category which is enriched over the stable model category of symmetric spectra) with a set of compact generators is Quillen-equivalent to dg-modules over a certain differential graded  $\mathbb{Q}$ -category. Thus every triangulated category equivalent to the homotopy category of a stable model category of this kind is algebraic. I think that the assumption ‘spectral’ is merely of a technical nature and will eventually be removed. Similarly, I expect that the assumption of a ‘set of compact generators’ can be relaxed to ‘well generated’ at the price of allowing localizations of module categories over a differential graded  $\mathbb{Q}$ -category, along the lines of the papers [Po] and [He]. At present, I do not know of a  $\mathbb{Q}$ -linear triangulated category which is not topological.

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MATHEMATISCHES INSTITUT, UNIVERSITÄT BONN, GERMANY  
*E-mail address:* [schwede@math.uni-bonn.de](mailto:schwede@math.uni-bonn.de)